PROBABILITY

Example Sheet 1

1. Let A, B, C be three events. Express in symbols the events:

(i) only A occurs

(ii) all three events occur

(iii) at least one event occurs

(iv) one and only one event occurs

(v) no event occurs

(vi) not more than two events occur.

2. From a table of random digits, k are chosen. What are the probabilities that for $0 \leqslant r \leqslant 9$,

(i) no digit exceeds r?

(ii) r is the greatest digit drawn?

3. Five mice are chosen (without replacement) from a litter, three of which are tagged A, B and C. The probability that all three tagged mice are chosen is twice the probability that A is the only tagged mouse chosen. How many mice are there in the litter?

4. (i) If A, B, C are three events, show that

$$P(A^c \cap (B \cup C)) = P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C).$$

(ii) How many of the numbers $1, \ldots, 500$ are not divisible by 7 but are divisible by 3 or 5?

5. A tennis championship is organized for 2^n players as a knock-out tournament with n rounds, the last round being the final. Two players are chosen at random. What are the probabilities that they meet

(i) in the first round?

(ii) in the final?

(iii) in any round of the tournament?

6. A sample of size r is taken from a population of size n, sampling without replacement. Calculate the probability that m given people will all be included in the sample (i) directly and (ii) by using the inclusion-exclusion formula. Hence show that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j}\binom{n-j}{r}.$$

7. Show that

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2.$$

8. An urn contains equal numbers of red balls and black balls. Suppose that a sample of $2n$ balls is chosen (with replacement) from the urn. Show that the probability that there are equal numbers of red balls and black balls in the sample is $(2n)!/[2^n(n!)]^2$.

Use Stirling's formula to show that this probability is approximately $1/\sqrt{\pi n}$ when n is large.

9. Two cards are taken at random from an ordinary pack of 52 cards. Find the probabilities that:

(i) both cards are aces (event A)

(ii) the pair of cards includes an ace (event B)

(iii) the pair of cards includes the ace of hearts (event C).

Show that $P(A|B) \neq P(A|C)$.

10. Examination candidates are graded into four classes known conventionally as I, II-1, II-2 and III, with probabilities 1/8, 2/8, 3/8 and 2/8 respectively. A candidate who misreads the rubric – a common event with probability 2/3 – generally does worse, his probabilities being 1/10, 2/10, 4/10 and 3/10. What is the probability:

(i) that a candidate who reads the rubric correctly is placed in the class II-1?

(ii) that a candidate who is placed in the class II-1 has read the rubric correctly?

11. Parliament contains a proportion p of Conservative members, who are incapable of changing their minds about anything, and a proportion $1 - p$ of Labour members who change their minds completely at random (with probability r) between successive votes on the same issue. A randomly chosen member is noticed to have voted twice in succession in the same way. What is the probability that he will vote in the same way next time?

Additional exercises:

12. There are n people gathered in a room. What is the probability that at least one has the same birthday as you? What value of n makes this probability close to $\frac{1}{2}$?

13. Suppose that n balls are placed at random into n boxes, find the probability that there is exactly one empty box.

14. A fair coin is tossed until either the sequence HHH occurs, in which case A wins, or the sequence THH occurs, when B wins. Calculate the probability B wins.

15. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets strictly more heads than John? Answer the same question if Mary tosses three coins and John tosses two. Make a conjecture for the same probability when Mary tosses $n+1$ coins and John tosses n. Can you prove your conjecture?

16. A total of n psychologists remembered to attend a meeting about absent-mindedness. After the meeting, none could recognise his own coat, so they took coats at random. Furthermore, each was liable, with probability p and independently of the others, to lose the coat on the way home. Assuming, optimistically, that all arrived home, show that the probability that none had his own coat with him is approximately $e^{-(1-p)}$.

17. You throw $6n$ dice at random. Show that the probability that each number appears exactly n times is

$$\frac{(6n)!}{(n!)^6}\left(\frac{1}{6}\right)^{6n}.$$

Use Stirling's formula ($n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}$) to show that this probability is approximately $cn^{-5/2}$ for some constant c to be determined.

Comments on this example sheet may be sent to: d.p.kennedy@statslab.cam.ac.uk

26 January 2010