DPK **Lent Term**

PROBABILITY

Example Sheet 2

1. A coin with probability p of heads is tossed n times. Let E be the event 'a head is obtained on the first toss' and F_k the event 'exactly k heads are obtained'. For which pairs of integers (n, k) are E and F_k independent?

2. The events A, B and C are independent. Show (i) that the events A^c, B and C are independent, (ii) that the events A^c, B^c and C are independent and (iii) that the events A^c, B^c and C^c are independent.

3. Independent trials are performed, each with probability p of success. Let P_n be the probability that n trials result in an even number of successes. Show that

$$P_n = \frac{1}{2}\big[1 + (1-2p)^n\big].$$

4. Two darts players A and B throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws A has probability p_A and B has probability p_B of scoring a bull. If A has first throw, calculate the probability of his winning the contest.

5. The number of mispsrints on a page has a Poisson distribution with parameter λ, and the numbers on different pages are independent. What is the probability that the second misprint will occur on page r?

6. Suppose that X and Y are independent random variables with the Poisson distribution with parameters λ and μ respectively. Prove that the conditional distribution of X, given that $X + Y = n$ is binomial with parameters n and $\lambda/(\lambda + \mu)$.

7. Suppose that $X_1, \ldots, X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of

$$S^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2, \text{ where } \overline{X} = \frac{1}{n}\sum_{1}^{n} X_i.$$

8. In a sequence of independent trials the probability of a success at the ith trial is p_i. Show that the mean and variance of the total number of successes are $n\overline{p}$ and

$n\overline{p}(1-\overline{p}) - \sum_i (p_i - \overline{p})^2$, where $\overline{p} = \sum_i p_i/n$. Notice that for a given mean, the variance is greatest when all the p_i are equal.

9. Let $(X,Y) = (\cos\theta, \sin\theta)$ where $\theta = K\pi/4$ and K is a random variable such that $\mathbb{P}(K = r) = \frac{1}{8}$, for $r = 0, 1, \ldots, 7$. Show that $\mathbb{C}\mathrm{ov}(X,Y) = 0$, but that X and Y are not independent.

10. Elmo's bowl of spaghetti contains n strands. He selects two ends at random and joins them together. He does this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

11. Julia collects figures from cornflakes packets. Each packet contains one figure, and n distinct figures make a complete set. Show that the expected number of packets that Julia needs to buy to collect a complete set is $n \sum_{i=1}^{n} 1/i$.

12. Suppose that $X_1, X_2, \ldots$ are independent identically distributed positive random variables with $\mathbb{E}\, X_1 = \mu < \infty$ and $\mathbb{E}\left(X_1^{-1}\right) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Show that $\mathbb{E}\,(S_m/S_n) = m/n$ when $m \leqslant n$, and $\mathbb{E}\,(S_m/S_n) = 1 + (m-n)\mu\mathbb{E}\left(S_n^{-1}\right)$, when $m \geqslant n$.

Additional exercises:

13. The probability that a football team will score n goals in a match is $p^n(1-p)$, $n = 0, 1, 2, \ldots$, independently of the performance of the other team. What is the probability of a score draw if teams with probabilities p_1, p_2 meet? If $p_1 = p_2 = p$, what value of p gives the highest probability of a score draw, and what is this probability?

(**Note:** A score draw is the situation where the final score is $n - n$, for $n \geqslant 1$; the situation of $0 - 0$ is a no-score draw.)

14. Let X be an integer-valued random variable with distribution

$$P(X = n) = n^{-s}/\zeta(s)$$

where $s > 1$, and $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$. Suppose that $p_1 < p_2 < p_3 < \cdots$ are the primes, and let A_k be the event $\{X \text{ is divisible by } p_k\}$. Find $\mathbb{P}(A_k)$ and show that the events $A_1, A_2, \ldots$ are independent. Deduce that

$$\prod_{k=1}^{\infty} (1 - p_k^{-s}) = 1/\zeta(s).$$

15. You are playing a match against an opponent in which at each point either you or your opponent serve. If you serve you win the point with probability p_1, but if your opponent serves you win the point with probability p_2. There are two possible conventions for serving:

(i) serves alternate

(ii) the player serving continues to serve until he loses a point.

You serve first and the first player to reach n points wins the match. Show that your probability of winning the match does not depend on the serving convention adopted. [**Hint:** under either convention you serve at most n times and your opponent at most $n-1$ times.]

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